



Classification of linearly compact simple Jordan and generalized Poisson superalgebras

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To Ernest Borisovich Vinberg on his 70th birthday

Abstract

We classify all linearly compact simple Jordan superalgebras over an algebraically closed field of characteristic zero. As a corollary, we deduce the classification of all linearly compact unital simple generalized Poisson superalgebras.

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0. Introduction

Jordan algebras first appeared in a 1933 paper by P. Jordan on axiomatic foundations of quantum mechanics. Subsequently, Jordan, von Neumann and Wigner classified the “totally positive” simple finite-dimensional Jordan algebras [9]. All simple finite-dimensional Jordan algebras over an algebraically closed field \mathbb{F} of characteristic different from 2 have been classified some years later by Albert [1], using the idempotent method.

Albert’s list is as follows:

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- A. $gl(m)_+$: the space of $m \times m$ -matrices over \mathbb{F} with multiplication $a \circ b = \frac{1}{2}(ab + ba)$;
- B. $o(m)_+$: the subalgebra of symmetric matrices in $gl(m)_+$;
- C. $sp(2m)_+$: the subalgebra of matrices in $gl(2m)_+$ fixed under the anti-involution $a \mapsto J^{-1}a^t J$, where J is a non-degenerate skew-symmetric matrix;
- D. $(m)_+$: $\mathbb{F}e \oplus \mathbb{F}^m$, where e is the unit element and the multiplication on \mathbb{F}^m is defined via a non-degenerate symmetric bilinear form (\cdot, \cdot) : $a \circ b = (a, b)e$;
- E. the 27-dimensional Jordan algebra E of Hermitian 3×3 -matrices over Cayley numbers.

Jordan, von Neumann and Wigner expressed a hope in their above cited paper that there should be new types of simple infinite-dimensional Jordan algebras, but in his remarkable paper [24] Zelmanov proved that this was not the case. Namely, all examples A_+ of type A can be obtained by replacing $m \times m$ -matrices by an infinite-dimensional simple associative algebra A ; all examples of types B and C can be obtained by taking the subalgebra of the Jordan algebra A_+ , fixed under an anti-involution of A ; all examples of type D are obtained by replacing \mathbb{F}^m by an infinite-dimensional vector space with a non-degenerate symmetric bilinear form; as a result, along with E , one gets a complete list of simple Jordan algebras over \mathbb{F} .

One of the advances of the theory of Jordan algebras in the 60s was the discovery of the Tits–Kantor–Koecher (TKK) construction [14,19,23]. It is based on the observation that if $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra with a *short* \mathbb{Z} -grading and $f \in \mathfrak{g}_1$, then the formula

$$a \circ b = [f, a], b \quad (0.1)$$

defines a structure of a Jordan algebra on \mathfrak{g}_{-1} . If $e \in \mathfrak{g}_{-1}$ is the unit element of this Jordan algebra and $h = [e, f]$ for some $f \in \mathfrak{g}_1$, then $\{e, h, f\}$ is an sl_2 -triple in \mathfrak{g} , and the eigenspace decomposition of $ad h$ defines a short grading of \mathfrak{g} . This leads to an equivalence of the category of unital Jordan algebras to the category of pairs $(\mathfrak{g}, \mathfrak{a})$, where \mathfrak{g} is a Lie algebra and \mathfrak{a} is its sl_2 subalgebra whose semisimple element with eigenvalues $0, 1, -1$ defines a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, such that \mathfrak{g}_0 contains no non-zero ideals of \mathfrak{g} and $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, the morphisms being epimorphisms. (Such \mathfrak{a} is called a short subalgebra of \mathfrak{g} .) The $(\mathbb{Z}$ -graded) Lie algebra corresponding to a Jordan algebra J under this equivalence is called the TKK construction of J . This construction can be applied to non-unital Jordan algebras as well, but then the corresponding category of \mathbb{Z} -graded Lie algebras is a bit more complicated [11]. However, it is easier to add the unit element to the Jordan algebra if it does not have one, which leads to a short subalgebra of the Lie algebra of outer derivations of the corresponding Lie algebra. In the present paper we adopt the latter approach.

Using the TKK construction, Kac [11] derived a classification of simple finite-dimensional Jordan superalgebras, under the assumption $\text{char } \mathbb{F} = 0$, from the classification of simple finite-dimensional Lie superalgebras [10]. The list consists of Jordan superalgebras of types A , BC and D , which are super analogues of the types A , B , C , D above, two “strange” series P and Q , the Kantor series [15], and, in addition to the exceptional Jordan algebra E , a family of 4-dimensional Jordan superalgebras D_t , a non-unital 3-dimensional Jordan superalgebra K , and an exceptional 10-dimensional Jordan superalgebra F . (Note that the multiplication table of F in [11] contained errors, which were fixed in [8], and that the Kantor series was inadvertently omitted in [11], as it has been pointed out in [15].)

It turned out that the Kantor series is a subseries of finite-dimensional Jordan superalgebras of a family of simple Jordan superalgebras, obtained via the Kantor–King–McCrimmon (KKM)

double, defined as follows. Let A be a commutative associative superalgebra, let $J(A) = A \oplus \eta A$, where η is an odd indeterminate, and define a product on $J(A)$ by $(a, b \in A)$:

$$a \circ b = ab, \quad \eta a \circ b = \eta(ab), \quad a \circ \eta b = (-1)^{p(a)} \eta(ab), \quad \eta a \circ \eta b = (-1)^{p(a)} \{a, b\}, \quad (0.2)$$

where $\{\cdot, \cdot\}$ is some other bilinear product on A . Kantor [15] checked that if $\{\cdot, \cdot\}$ is a Poisson bracket, i.e., a Lie superalgebra bracket satisfying the Leibniz rule, then $J(A)$ is a Jordan superalgebra. For example, the Grassmann superalgebra $\Lambda(n) = \Lambda(\xi_1, \dots, \xi_n)$ with bracket [10]

$$\{f, g\} = (-1)^{p(f)} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i} \quad (0.3)$$

is a Poisson superalgebra. Hence $JP(0, n) = \Lambda(n) \oplus \eta \Lambda(n)$ with product (0.2) is a Jordan superalgebra, and this is the Kantor series, missed in [11].

King and McCrimmon [16] found necessary and sufficient conditions on the bracket $\{\cdot, \cdot\}$ in order for (0.2) to give a Jordan product. Using this, we show that all the KKM doubles $J(A)$, which are Jordan superalgebras, can be obtained from a generalized Poisson bracket on the superalgebra A . By definition, this is a Lie superalgebra bracket $\{\cdot, \cdot\}$, satisfying the generalized Leibniz rule

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)} b\{a, c\} + D(a)bc, \quad (0.4)$$

where D is an even derivation of the product and the bracket. (If $D = 0$ we get the usual Leibniz rule, and if A has a unit element e , then $D(a) = \{e, a\}$.)

Now, the commutative associative superalgebra $\mathcal{O}(m, n) = \Lambda(n)[\llbracket x_1, \dots, x_m \rrbracket]$ carries a well known structure of a generalized Poisson superalgebra [10,12], which we denote by $P(m, n)$. The KKM double of this is a simple (unless $m = n = 0$) Jordan superalgebra, denoted by $JP(m, n)$. The case when $m = 1$ and $\mathcal{O}(1, n)$ is replaced by $\Lambda(n)[x, x^{-1}]$ plays a prominent role in the Kac–Martinez–Zelmanov (KMZ) classification [13]; denote this Jordan superalgebra by $JP(1, n)^\sim$. It is shown in [13], that $JP(1, 3)^\sim$ contains a “half size” exceptional Jordan superalgebra JCK^\sim , whose TKK construction is the exceptional Lie superalgebra CK_6 , discovered by Cheng and Kac [5]. The KMZ classification theorem [13] states that a complete list of \mathbb{Z} -graded simple unital Jordan superalgebras of growth 1 consists of twisted loop algebras over finite-dimensional simple Jordan superalgebras, Jordan superalgebras of a non-degenerate supersymmetric bilinear form on a \mathbb{Z} -graded vector superspace (type D above), the series $JP(1, n)^\sim$, JCK^\sim and their twisted analogues, and simple Jordan superalgebras of “Cartan type.”

By definition, a Lie superalgebra L is called of Cartan type if it has a subalgebra L_0 of finite codimension, containing no non-zero ideals of L ; a Jordan superalgebra is of Cartan type if it contains a subalgebra of finite codimension and its TKK construction is of Cartan type. One can construct a descending filtration of a Lie superalgebra of Cartan type $L \supset L_0 \supset L_1 \supset L_2 \supset \dots$ (see e.g. [12]), introduce a structure of a topological Lie superalgebra on L by taking $\{L_j\}_{j \in \mathbb{Z}_+}$ to be the fundamental system of neighbourhoods of zero, and consider the completion \bar{L} of L with respect to this topology. The topological Lie superalgebra \bar{L} is linearly compact in the sense of the following definition.

A topological superalgebra (associative, or Lie, or Jordan) is called linearly compact if the underlying topological space is linearly compact (i.e., it is isomorphic to the topological direct product of finite-dimensional vector spaces with the discrete topology). It is well known (see [7]) that any linearly compact Lie superalgebra has a subalgebra of finite codimension, hence any linearly compact Lie superalgebra is of Cartan type.

Infinite-dimensional simple linearly compact Lie superalgebras were classified by Kac [12]. The main goal of the present paper is to establish the equivalence (similar to the discussed above) of the category of unital linearly compact Jordan superalgebras and the category of linearly compact Lie superalgebras with a short subalgebra, and to use this equivalence in order to derive the classification of infinite-dimensional linearly compact simple Jordan superalgebras from the classification in the Lie superalgebra case.

It turns out that among the ten series of infinite-dimensional linearly compact simple Lie superalgebras only the series $H(m, n)$ and $K(m, n)$ admit a (unique) short subalgebra (provided that $n \geq 3$), and the superalgebra $S(1, 2)$ admits a short algebra of outer derivations. In addition, among the five exceptional ones, only $E(1, 6)$ admits a (unique) short subalgebra. As a result, we obtain the following list of all infinite-dimensional linearly compact simple Jordan superalgebras: the series of unital Jordan superalgebras $JP(m, n)$ (mentioned above), the exceptional unital Jordan superalgebra JCK (which is a “half-size” subalgebra of $JP(1, 3)$), and a non-unital Jordan superalgebra, which we denote by JS .

The construction of the latter is very simple. Let $JS = \mathcal{O}(1, 1)$ with reversed parity, and the product

$$a \circ b = aD(b) + (-1)^{p(a)}D(a)b, \quad (0.5)$$

where $D = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}$.

Along the way, we redo the (corrected) classification of [11], using the simpler method of short subalgebras.

As a corollary, we obtain that any simple unital generalized Poisson superalgebra is isomorphic to one of the $P(m, n)^\varphi$, where $P(m, n)$ is one of the “standard” generalized Poisson superalgebras, mentioned above, and $P(m, n)^\varphi$ is obtained from it by twisting the bracket by an even invertible element $\varphi \in \mathcal{O}(m, n)$ by the formula

$$\{a, b\}^\varphi = \varphi^{-1}\{\varphi a, \varphi b\}. \quad (0.6)$$

We hope that the blow to Jordan, von Neumann, Wigner expectations, dealt by Zelmanov’s theorem on non-existence of new types of simple infinite-dimensional Jordan algebras, will be remedied by applications of simple infinite-dimensional Jordan superalgebras to supersymmetric quantum mechanics.

Throughout the paper the base field \mathbb{F} is algebraically closed of characteristic 0, unless otherwise specified.

1. Preliminaries on superalgebras

We recall that a vector *superspace* V is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space: $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If a lies in V_α , $\alpha \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, we say that α is the *parity* of a and we denote it by $p(a)$. The elements of $V_{\bar{0}}$ are called *even*, the elements of $V_{\bar{1}}$ are called *odd*.

A bilinear form (\cdot, \cdot) on a vector superspace V is called *supersymmetric* (respectively *superskew-symmetric*) if its restriction to $V_{\bar{0}}$ is symmetric (respectively skew-symmetric), its restriction to $V_{\bar{1}}$ is skew-symmetric (respectively symmetric) and $(V_{\bar{0}}, V_{\bar{1}}) = 0$. The matrix of a supersymmetric (respectively superskew-symmetric) bilinear form on a vector superspace V in some basis of V , is called supersymmetric (respectively superskew-symmetric).

A *superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. A superalgebra A is called *commutative* if, for every $a, b \in A$,

$$ab = (-1)^{p(a)p(b)}ba.$$

Associativity of superalgebras is defined as for algebras.

Given a vector superspace V , the associative algebra $\text{End}(V)$ is equipped with the induced $\mathbb{Z}/2\mathbb{Z}$ -grading: $\text{End}(V) = \bigoplus_{\alpha \in \mathbb{Z}/2\mathbb{Z}} \text{End}_{\alpha}(V)$, where

$$\text{End}_{\alpha}(V) = \{f \in \text{End}(V) \mid f(V_s) \subseteq V_{s+\alpha}\}.$$

The commutator in an associative superalgebra is defined by

$$[a, b] = ab - (-1)^{p(a)p(b)}ba.$$

With respect to this bracket, $\text{End}(V)$ is a Lie superalgebra.

A derivation with parity s , $s \in \mathbb{Z}/2\mathbb{Z}$, of a superalgebra A is an endomorphism $D \in \text{End}_s(A)$ with the property

$$D(ab) = D(a)b + (-1)^{p(a)s}aD(b).$$

The space $\text{Der } A$ of all derivations of A is a subalgebra of the Lie superalgebra $\text{End}(A)$.

A super anti-involution of a superalgebra A is a grading preserving linear map $^*: A \rightarrow A$ such that $(a^*)^* = a$ for every $a \in A$, and

$$(ab)^* = (-1)^{p(a)p(b)}b^*a^* \quad (a, b \in A).$$

If (\cdot, \cdot) is a non-degenerate supersymmetric bilinear form on a vector superspace V of dimension $(m|n)$, then we have a super anti-involution $^*: \text{End}(V) \rightarrow \text{End}(V)$, defined by

$$(f(a), b) = (-1)^{p(a)p(f)}(a, f^*(b)), \quad f \in \text{End}(V), \quad a, b \in V. \quad (1.1)$$

If B is the matrix of this bilinear form in some basis of V , compatible with grading, and M (respectively M^*) is the matrix of an endomorphism f (respectively f^*) in the same basis, then

$$a^t M^t B b = (-1)^{p(a)p(M)} a^t B M^* b, \quad (1.2)$$

where $^*: \text{Mat}(m, n) \rightarrow \text{Mat}(m, n)$ is a super anti-involution given by $M^* = B^{-1} M^T B$, and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} \alpha^t & \gamma^t \\ -\beta^t & \delta^t \end{pmatrix},$$

with t standing for the usual transpose.

If (\cdot, \cdot) is an odd supersymmetric bilinear form, i.e., its matrix in some basis of V , compatible with grading, is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then the super anti-involution $*$, defined in (1.1), in matrix form looks as (1.2), where $*$: $Mat(m, m) \rightarrow Mat(m, m)$ is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta^t & \beta^t \\ -\gamma^t & \alpha^t \end{pmatrix}.$$

Given a vector superspace V , a bilinear map $\varphi: V \times V \rightarrow V$ is called commutative if $\varphi(a, b) = (-1)^{p(a)p(b)}\varphi(b, a)$; it is called anticommutative if $\varphi(a, b) = -(-1)^{p(a)p(b)}\varphi(b, a)$.

2. Preliminaries on linearly compact spaces and algebras

Recall that a *linear topology* on a vector space V over \mathbb{F} is a topology for which there exists a fundamental system of neighbourhoods of zero, consisting of vector subspaces of V . A vector space V with linear topology is called *linearly compact* if there exists a fundamental system of neighbourhoods of zero, consisting of subspaces of finite codimension in V and, in addition, V is complete in this topology (equivalently, if V is a topological direct product of some number of copies of \mathbb{F} with discrete topology).

If $\varphi: V \rightarrow U$ is a continuous map of vector spaces with linear topology and V is linearly compact, then $\varphi(V)$ is linearly compact [7].

The basic examples of linearly compact spaces are finite-dimensional vector spaces with the discrete topology, and the space of formal power series $V[[x_1, \dots, x_m]]$ over a finite-dimensional vector space V , with the topology defined by taking as a fundamental system of neighbourhoods of 0 the subspaces $\{x_1^{j_1} \dots x_m^{j_m} V[[x_1, \dots, x_m]]\}_{(j_1, \dots, j_m) \in \mathbb{Z}_+^m}$.

Let U and V be two vector spaces with linear topology, and let $Hom(U, V)$ denote the space of all continuous linear maps from U to V . We endow the vector space $Hom(U, V)$ with a “compact-open” linear topology, the fundamental system of neighbourhoods of zero being $\{\Omega_{K,W}\}$, where K runs over all linearly compact subspaces of U , W runs over all open subspaces of V , and

$$\Omega_{K,W} = \{\varphi \in Hom(U, V) \mid \varphi(K) \subset W\}.$$

In particular, we define the dual space of V as $V^* = Hom(V, \mathbb{F})$, where \mathbb{F} is endowed with the discrete topology. It is easy to see that V is linearly compact if and only if V^* is discrete. Note also that a discrete space V is linearly compact if and only if $\dim V < \infty$.

The tensor product of two vector spaces U and V with linear topology is defined as

$$U \hat{\otimes} V = Hom(U^*, V).$$

Thus, $U \hat{\otimes} V = U \otimes V$ if both U and V are discrete and $U \hat{\otimes} V = (U^* \otimes V^*)^*$ if both U and V are linearly compact. Hence the tensor product of linearly compact spaces is linearly compact [7].

Recall that the basic property of the “compact-open” topology on $Hom(U, V)$ is the following canonical isomorphism of vector spaces with linear topology:

$$Hom(M, Hom(U, V)) \simeq Hom(M \hat{\otimes} U, V). \quad (2.1)$$

(This isomorphism is the closure of the map $\Phi(\psi)(m \otimes u) = \psi(m)(u)$.)

A *linearly compact superalgebra* is a topological superalgebra whose underlying topological space is linearly compact. Of course, any finite-dimensional superalgebra is linearly compact. The basic example of an associative linearly compact superalgebra is the superalgebra $\mathcal{O}(m, n) = \Lambda(n)[[x_1, \dots, x_m]]$, where $\Lambda(n)$ denotes the Grassmann algebra on n anticommuting indeterminates ξ_1, \dots, ξ_n , and the superalgebra parity is defined by $p(x_i) = \bar{0}$, $p(\xi_j) = \bar{1}$, with the formal topology defined as above for $V = \Lambda(n)$. The basic example of a linearly compact Lie superalgebra is $W(m, n) = \text{Der } \mathcal{O}(m, n)$, the Lie superalgebra of all continuous derivations of the superalgebra $\mathcal{O}(m, n)$. One has

$$W(m, n) := \left\{ X = \sum_{i=1}^m P_i(x, \xi) \frac{\partial}{\partial x_i} + \sum_{j=1}^n Q_j(x, \xi) \frac{\partial}{\partial \xi_j} \mid P_i, Q_j \in \mathcal{O}(m, n) \right\}.$$

Lemma 2.1. *Let A be a linearly compact (super)algebra, and let L_a denote the operator of left multiplication by a . Then:*

- (a) *the subspace $\{L_a \mid a \in A\}$ of the space of all continuous linear maps $\text{Hom}(A, A)$ is linearly compact;*
- (b) *the closure of the set $[L_a, L_b]$ in $\text{Hom}(A, A)$ is linearly compact.*

Proof. We apply (2.1) to $M = A$ (respectively $M = A \hat{\otimes} A$) and $U = V = A$ for the proof of (a) (respectively (b)). It follows that the map $a \mapsto L_a$ is continuous, and since the image of a linearly compact space under a continuous map is linearly compact, (a) follows. The proof of (b) is similar by making use of the fact that $A \hat{\otimes} A$ is linearly compact. \square

3. Generalized Poisson superalgebras and the Schouten bracket

Definition 3.1. A generalized Poisson superalgebra is a superalgebra A with two operations: a commutative associative superalgebra product $(a, b) \mapsto ab$ and a Lie superalgebra bracket $(a, b) \mapsto \{a, b\}$, jointly satisfying the generalized Leibniz rule, namely, for $a, b, c \in A$, one has

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)}b\{a, c\} + D(a)bc, \quad (3.1)$$

for some even derivation D of A with respect to the product and the bracket. If $D = 0$, then relation (3.1) becomes the usual Leibniz rule; in this case A is called a Poisson superalgebra. A generalized Poisson superalgebra is called simple if it contains no non-trivial ideals with respect to both operations and the bracket is non-trivial.

Remark 3.2. If A is a unital generalized Poisson superalgebra and e is its unit element, then $D(a) = \{e, a\}$. Indeed, it is sufficient to set $b = c = e$ in (3.1).

Example 3.3. Consider the associative superalgebra $\mathcal{O}(2k, n)$ with even indeterminates $p_1, \dots, p_k, q_1, \dots, q_k$, and odd indeterminates ξ_1, \dots, ξ_n . Define on $\mathcal{O}(2k, n)$ the following bracket $(f, g \in \mathcal{O}(2k, n))$:

$$\{f, g\} = \sum_{i=1}^k \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) + (-1)^{p(f)} \left(\sum_{i=1}^{n-2} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i} + \frac{\partial f}{\partial \xi_{n-1}} \frac{\partial g}{\partial \xi_n} + \frac{\partial f}{\partial \xi_n} \frac{\partial g}{\partial \xi_{n-1}} \right). \quad (3.2)$$

Then $\mathcal{O}(2k, n)$ with this bracket is a Poisson superalgebra (cf. [6, §1.2]), which we denote by $P(2k, n)$.

Consider $\mathcal{O}(2k, n)$ with its Lie superalgebra structure defined by bracket (3.2). Then $\mathbb{F}1$ is a central ideal, hence $H'(2k, n) := \mathcal{O}(2k, n)/\mathbb{F}1$ is a linearly compact Lie superalgebra [6]. Set $H(2k, n) = [H'(2k, n), H'(2k, n)]$. We recall that $H(2k, n) = H'(2k, n)$ if $k \geq 1$, and then it is a simple Lie superalgebra; besides, $H(0, n)$ is simple if and only if $n \geq 4$ and $H'(0, n) = H(0, n) + \mathbb{F}\xi_1 \dots \xi_n$ (see [11, Proposition 3.3.6], [10, Example 4.3]).

Example 3.4. Consider the associative superalgebra $\mathcal{O}(2k+1, n)$ with even indeterminates $t, p_1, \dots, p_k, q_1, \dots, q_k$, and odd indeterminates ξ_1, \dots, ξ_n . Define on $\mathcal{O}(2k+1, n)$ the following bracket ($f, g \in \mathcal{O}(2k+1, n)$):

$$\{f, g\}_K = (2-E)f \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)g + \{f, g\}, \quad (3.3)$$

where $E = \sum_{i=1}^k (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}) + \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}$ is the Euler operator and $\{f, g\}$ is given by (3.2). Then $\mathcal{O}(2k+1, n)$ with bracket $\{\cdot, \cdot\}_K$ is a generalized Poisson superalgebra with $D = 2 \frac{\partial}{\partial t}$ (cf. [3, Remark 2.19]), which we denote by $P(2k+1, n)$.

We shall denote by $K(2k+1, n)$ the superalgebra $P(2k+1, n)$ with its Lie superalgebra structure. This is a simple linearly compact Lie superalgebra for every k and n (cf. [6, §1.2]).

Remark 3.5. Let $C = (f_{ij})$ be a $(2k+n) \times (2k+n)$ superskew-symmetric matrix over \mathbb{F} . Then a Poisson bracket can be defined on $\mathcal{O}(2k, n)$ as follows: denote by X_i , with $i = 1, \dots, 2k$ and $i = 2k+1, \dots, 2k+n$, all even and odd indeterminates, respectively. Define

$$\{X_i, X_j\} = f_{ij}, \quad (3.4)$$

and then extend $\{\cdot, \cdot\}$ to all polynomials by the usual Leibniz rule. Next, consider the associative superalgebra $\mathcal{O}(2k+1, n)$ with the same indeterminates as above and one more even indeterminate t . Then a generalized Poisson bracket, with $D = 2 \frac{\partial}{\partial t}$, can be defined on $\mathcal{O}(2k+1, n)$ by (3.4) and

$$\{t, X_i\} = -X_i, \quad (3.5)$$

extending by the generalized Leibniz rule (3.1). Note that Examples 3.3 and 3.4 are isomorphic to these with a non-degenerate matrix C , by an invertible linear change of indeterminates. Note also that all generalized Poisson superalgebras $P(m, n)$, except for $P(0, 0)$, are simple.

Now we briefly discuss the connection of generalized Poisson superalgebras with the Schouten bracket.

Let A be a commutative associative superalgebra over \mathbb{F} . Then $\text{Der } A$ is a Lie superalgebra over \mathbb{F} and a left A -module. Define on $\Lambda(\text{Der } A)$, the exterior algebra over $\text{Der } A$, viewed as a module over A , the following parity: $p(fX_1 \wedge \dots \wedge X_k) = p(f) + p(X_1) + \dots + p(X_k) + k$,

for $f \in A$ and $X_i \in \text{Der } A$, $i = 1, \dots, k$. The Schouten bracket $[\cdot, \cdot]_S$ on $\Lambda(\text{Der } A)$ is defined as follows:

$$\begin{aligned} [f, g]_S &= 0 \quad \text{for } f, g \in A; & [X, Y]_S &= [X, Y] \quad \text{for } X, Y \in \text{Der } A; \\ [D, f]_S &= -(-1)^{(p(D)+1)(p(f)+1)}[f, D]_S = D(f) \quad \text{for } D \in \text{Der } A, f \in A; \end{aligned}$$

and extended by the odd Leibniz rule:

$$[a, b \wedge c]_S = [a, b]_S \wedge c + (-1)^{(p(a)+1)p(b)} b \wedge [a, c]_S.$$

Then $[\cdot, \cdot]_S$ satisfies the Lie superalgebra axioms after reversing parity.

Example 3.6. If $A = \mathbb{F}[[x_1, \dots, x_n]]$, then the Schouten bracket is the Buttin bracket (see [6]).

Lemma 3.7. (Cf. [17,18].) Let A be a commutative associative superalgebra, a an even derivation of A , and $c = \sum_i b_i \wedge d_i$ an even element in $\Lambda^2(\text{Der } A)$. Then the bracket

$$\{f, g\} = fa(g) - a(f)g + \sum_i (-1)^{p(f)p(b_i)} (b_i(f)d_i(g) - (-1)^{p(b_i)} d_i(f)b_i(g)) \quad (3.6)$$

satisfies the Jacobi identity if and only if

$$[a, c]_S = 0 \quad \text{and} \quad [c, c]_S = -2a \wedge c, \quad (3.7)$$

where $[\cdot, \cdot]_S$ is the Schouten bracket on $\Lambda(\text{Der } A)$.

Examples 3.8. (H) Let $A = \mathcal{O}(2k, n)$, $a = 0$ and $c = \sum_{i,j} f_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, for some superskew-symmetric matrix (f_{ij}) over \mathbb{F} . Then conditions (3.7) hold, hence the corresponding bracket (3.6) defines a Poisson superalgebra structure on A .

(K) Let $A = \mathcal{O}(2k+1, n)$, with even indeterminates t, z_1, \dots, z_{2k} , and odd indeterminates $z_{2k+1}, \dots, z_{2k+n}$. If $a = 2 \frac{\partial}{\partial t}$ and $c = \sum_{i,j} f_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} - E \wedge \frac{\partial}{\partial t}$, where (f_{ij}) is a superskew-symmetric matrix over \mathbb{F} and $E = \sum_{i=1}^{2k+n} z_i \frac{\partial}{\partial z_i}$ is the Euler operator, then conditions (3.7) hold. It follows that A with bracket (3.6) is a generalized Poisson superalgebra.

Examples 3.3 and 3.4 are particular cases of Examples (H) and (K), respectively, when the matrix (f_{ij}) is invertible.

Example 3.9. Let A be a unital generalized Poisson superalgebra with bracket $\{\cdot, \cdot\}$ and derivation D . Then for any even invertible element $\varphi \in A$, formula

$$\{f, g\}^\varphi = \varphi^{-1} \{\varphi f, \varphi g\}$$

defines another generalized Poisson bracket on the superalgebra A , the corresponding derivation being $D^\varphi(a) = D(\varphi)a + \{\varphi, a\}$. We denote this generalized Poisson superalgebra by A^φ . The generalized Poisson superalgebras A and A^φ are called gauge equivalent.

4. Examples of Jordan superalgebras

Definition 4.1. A Jordan superalgebra is a commutative superalgebra J whose product $(a, b) \mapsto a \circ b$ satisfies the following identity:

$$(-1)^{p(a)p(c)}[L_{a \circ b}, L_c] + (-1)^{p(b)p(a)}[L_{b \circ c}, L_a] + (-1)^{p(c)p(b)}[L_{c \circ a}, L_b] = 0, \quad (4.1)$$

where $L_a \in \text{End}(J)$ denotes the operator of left multiplication by a : $L_a(x) = a \circ x$.

Remark 4.2. The following important relation is proved for Jordan superalgebras in the same way as for Jordan algebras [1]:

$$[[L_a, L_b], L_c] = (-1)^{p(b)p(c)} L_{a \circ (c \circ b) - (a \circ c) \circ b}. \quad (4.2)$$

Now we list all the examples of simple finite-dimensional Jordan superalgebras (see [11, §1.3]). We denote by $gl(m, n)_+$ the full linear Jordan superalgebra, i.e., the Jordan superalgebra of endomorphisms of a vector superspace V of dimension $(m|n)$, with product $a \circ b = \frac{1}{2}(ab + (-1)^{p(a)p(b)}ba)$. We denote by $osp(m, n)_+$ the subalgebra of $gl(m, n)_+$ consisting of self-adjoint endomorphisms with respect to a non-degenerate supersymmetric bilinear form (\cdot, \cdot) . We denote by $(m, n)_+$ the Jordan superalgebra defined on $\mathbb{F}e \oplus V$, where V is an $(m|n)$ -dimensional vector superspace with a supersymmetric bilinear form (\cdot, \cdot) , e is the unit element, and the product on V is defined by $a \circ b = (a, b)e$. We denote by $p(n)_+$ the subalgebra of $gl(n, n)_+$ consisting of selfadjoint operators with respect to a non-degenerate odd supersymmetric form; finally, we denote by $q(n)_+$ the subalgebra of all operators in $gl(n, n)_+$, commuting with the operator $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. The series $JP(0, n)$ was defined in the introduction. Finally, for the definition of D_t , F and K we refer to Appendix B. The dimensions of these Jordan superalgebras J are listed in the following table:

J	$(\dim J_{\bar{0}} \dim J_{\bar{1}})$
$gl(m, n)_+$	$(m^2 + n^2 2mn)$
$osp(m, n)_+$	$(\frac{1}{2}m(m+1) + \frac{1}{2}n(n-1) mn)$
$(m, n)_+$	$(1 + m n)$
$p(n)_+$	$(n^2 n^2)$
$q(n)_+$	$(n^2 n^2)$
D_t	$(2 2)$
E	$(27 0)$
F	$(6 4)$
K	$(1 2)$
$JP(0, n)$	$(2^n 2^n)$

Next, we construct some examples of infinite-dimensional Jordan superalgebras.

Example 4.3. Let U be a commutative associative superalgebra together with an anticommutative bilinear operation $\{\cdot, \cdot\}$, and consider a direct sum of vector superspaces $J = J(U, \{\cdot, \cdot\}) = U + \eta U$ where the symbol η has odd parity, i.e., the $\mathbb{Z}/2\mathbb{Z}$ -grading on U is extended to J by setting $J_{\bar{0}} = U_{\bar{0}} + \eta U_{\bar{1}}$, $J_{\bar{1}} = U_{\bar{1}} + \eta U_{\bar{0}}$. We define a multiplication on J as follows: for arbitrary elements $a, b \in U$, their product in J is the product ab in U , and $a(\eta b) = (-1)^{p(a)}\eta(ab)$,

$(\eta a)b = \eta(ab)$, $(\eta a)(\eta b) = (-1)^{p(a)}\{a, b\}$. Then $J(U, \{\cdot, \cdot\})$ is a superalgebra called the Kantor–King–McCrimmon (KKM) double of U .

Let A be a generalized Poisson superalgebra with bracket $\{\cdot, \cdot\}$ and derivation D , and define on A the following new bracket:

$$\{f, g\}_D := \{f, g\} - \frac{1}{2}(fD(g) - D(f)g), \quad f, g \in A.$$

Set $D' := \frac{1}{2}D$. Then the following identities hold:

$$\{f, gh\}_D = \{f, g\}_D h + (-1)^{p(f)p(g)} g\{f, h\}_D + D'(f)gh; \quad (4.3)$$

$$\begin{aligned} \{f, \{g, h\}_D\}_D &= \{\{f, g\}_D, h\}_D + (-1)^{p(f)p(g)} \{g, \{f, h\}_D\}_D - D'(f)\{g, h\}_D \\ &\quad - (-1)^{p(f)(p(g)+p(h))} D'(g)\{h, f\}_D - (-1)^{p(h)(p(f)+p(g))} D'(h)\{f, g\}_D. \end{aligned} \quad (4.4)$$

By [16, Theorem 2], the KKM double $J(A, \{\cdot, \cdot\}_D)$ is a Jordan superalgebra. We will denote it simply by $J(A)$. This result for Poisson superalgebras was obtained by Kantor [15, Theorem 3.1]: notice that if A is a Poisson superalgebra, then $\{\cdot, \cdot\}_D = \{\cdot, \cdot\}$.

We set $JP(m, n) := J(P(m, n))$. If $m = 0$, this is the Kantor series of Jordan superalgebras [15]. All the other Jordan superalgebras $JP(m, n)$ are infinite-dimensional linearly compact.

Remark 4.4. If $(A, \{\cdot, \cdot\})$ is unital, then it is simple if and only if $(A, \{\cdot, \cdot\}_D)$ is simple. This follows from Remark 3.2.

Example 4.5. Let A be an associative commutative algebra over \mathbb{F} with a derivation D , and let $\mathbb{H} = \mathbb{F}1 \oplus \mathbb{H}_{im}$ be the algebra of degenerate quaternions over \mathbb{F} , i.e., 1 is the unit element and $\mathbb{H}_{im} = \mathbb{F}i \oplus \mathbb{F}j \oplus \mathbb{F}k$ is a 3-dimensional associative algebra with the product $i \times j = k$, $j \times k = i$, $k \times i = j$, $i^2 = j^2 = k^2 = 0$. Let (\cdot, \cdot) be a symmetric bilinear form on \mathbb{H}_{im} , with respect to which $\{i, j, k\}$ is an orthonormal basis, and extend both the product \times and the form (\cdot, \cdot) to $A \otimes \mathbb{H}_{im}$ by bilinearity.

The Jordan superalgebra $JCK(A, D)$ is defined as the commutative superalgebra with even part $A \otimes \mathbb{H}$ and odd part $\eta(A \otimes \mathbb{H})$, where η is an odd indeterminate, and the following product (cf. [13, 20]):

$$\begin{array}{ll} f \circ g = fg & \text{if } f \in A \otimes 1, g \in A \otimes \mathbb{H}; \\ f \circ g = (f, g)1 & \text{if } f, g \in A \otimes \mathbb{H}_{im}; \\ \eta f \circ g = \eta(fg) & \text{if } g \in A \otimes 1; \\ \eta f \circ g = \eta(fD(g)) & \text{if } f \in A \otimes 1, g \in A \otimes \mathbb{H}_{im}; \\ \eta f \circ g = \sqrt{-1}\eta(f \times g) & \text{if } f, g \in A \otimes \mathbb{H}_{im}; \\ \eta f \circ \eta g = D(f)g - fD(g) & \text{if } f, g \in A \otimes 1; \\ \eta f \circ \eta g = fg & \text{if } f \in A \otimes \mathbb{H}_{im}, g \in A \otimes 1; \\ \eta f \circ \eta g = 0 & \text{if } f, g \in A \otimes \mathbb{H}_{im}. \end{array}$$

This Jordan superalgebra is simple if and only if A contains no non-trivial D -invariant ideals. If $A = \mathbb{C}[[x]]$ and $D = \frac{d}{dx}$, we will denote $JCK(A, D)$ simply by JCK .

Example 4.6. Let A be a commutative associative superalgebra with an odd derivation D . Set $J = A$ with reversed parity and define on J the following product:

$$a \circ b = aD(b) + (-1)^{p(a)p(b)}bD(a), \quad a, b \in J.$$

A rather long, but straightforward, calculation shows that (J, \circ) is a Jordan superalgebra, which will be denoted by $JS(A, D)$. If $A = \mathcal{O}(1, 1)$ and $D = \xi \frac{\partial}{\partial x} + \frac{\partial}{\partial \xi}$, we will denote $JS(A, D)$ simply by JS .

This construction is a generalization of the “half-unital double” $J(A, d)$, studied in [21] and [25], defined as follows. Let A be a commutative associative algebra with a derivation d . Then $J(A, d) = A + \eta A$ with product

$$a \circ b = ab, \quad \eta a \circ b = a \circ \eta b = \frac{1}{2}\eta ab, \quad \eta a \circ \eta b = d(a)b - ad(b).$$

Note that $J(A, d) \simeq JS(A[\eta], \frac{d}{d\eta} + \eta d)$, in particular $JS \simeq J(\mathbb{F}[[x]], \frac{d}{dx})$.

In Section 7 we will prove the following theorem:

Theorem 4.7. *All linearly compact simple Jordan superalgebras over \mathbb{F} are, up to isomorphism, the following:*

1. *The finite-dimensional Jordan superalgebras $gl(m, n)_+$, $osp(m, 2r)_+$ with $(m, r) \neq (0, 1)$, $(m, 2r)_+$ with $(m, r) \neq (1, 0)$, $p(n)_+$ with $n > 1$, $q(n)_+$ with $n > 1$, $JP(0, n)$ with $n \geq 1$, D_t with $t \in \mathbb{F}$, $t \neq 0$, E , F , K .*
2. *The infinite-dimensional Jordan superalgebras $JP(m, n)$ with $m \geq 1$, JCK , and JS .*

Remark 4.8. It follows from [10, §4.2.2] and the multiplication table of the Jordan superalgebras $(m, n)_+$ and D_t , that the following isomorphisms hold:

$$JP(0, 1) \simeq gl(1, 1)_+, \quad (1, 2)_+ \simeq D_1, \quad D_t \simeq D_{t-1}.$$

These are the only isomorphisms of Jordan superalgebras, listed in Theorem 4.7.

Remark 4.9. The Jordan superalgebra $JP(m, n)$ has growth m and size 2^{n+1} ; the Jordan superalgebras JCK and JS have growth 1 and sizes 8 and 2, respectively. (For the definition of growth and size, see [3, §1].) Note that in all cases the sizes of the even and the odd parts of J are equal to $\frac{1}{2}$ size (J) (of course, their growths are both equal to the growth of J).

5. TKK construction

Let J be a Jordan superalgebra and let P be the commutative bilinear map on J defined by: $P(x, y) = x \circ y$. Notice that the $\mathbb{Z}/2\mathbb{Z}$ -grading on J induces a natural $\mathbb{Z}/2\mathbb{Z}$ -grading on the spaces of linear and bilinear functions on J with values in J . We associate to J a \mathbb{Z} -graded Lie superalgebra $Lie(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as follows (cf. [11, 14, 19, 23]). We set: $\mathfrak{g}_{-1} = J$, $\mathfrak{g}_0 = \langle L_a, [L_a, L_b] \mid a, b \in J \rangle \subset End(\mathfrak{g}_{-1})$, $\mathfrak{g}_1 = \langle P, [L_a, P] \mid a \in J \rangle$ (the bracket of a linear and a bilinear map is defined below), and we define the following anticommutative brackets:

- $[x, y] = 0$ for $x, y \in \mathfrak{g}_{-1}$ or $x, y \in \mathfrak{g}_1$;
- $[a, x] = a(x)$ for $a \in \mathfrak{g}_0, x \in \mathfrak{g}_{-1}$;
- $[A, x](y) = A(x, y)$ for $A \in \mathfrak{g}_1, x, y \in \mathfrak{g}_{-1}$;
- $[a, B] = a \square B - (-1)^{p(a)p(B)} B \square a$ for $a \in \mathfrak{g}_0, B \in \mathfrak{g}_1$, where
 - (i) $(a \square B)(x, y) = a(B(x, y))$,
 - (ii) $(B \square a)(x, y) = B(a(x), y) + (-1)^{p(x)p(y)} B(a(y), x)$.

$Lie(J)$ is called the Tits–Kantor–Koecher (TKK) construction for J .

Proposition 5.1. *$Lie(J)$ is a Lie superalgebra.*

Proof. Using the definition of the bracket on $Lie(J)$ one shows that the following relations hold:

$$[P, x] = L_x, \quad x \in J; \quad (5.1)$$

$$[[L_a, P], x] = [L_a, L_x] - L_{a \circ x}, \quad a, x \in J. \quad (5.2)$$

It follows, using (5.2) and (4.2), that:

$$[L_a, [L_b, P]] = -[L_{a \circ b}, P], \quad a, b \in J; \quad (5.3)$$

hence, due to the commutativity of the Jordan product,

$$[[L_a, L_b], P] = 0, \quad a, b \in J. \quad (5.4)$$

Relations (5.1) and (5.2) show that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$. Besides, a direct computation shows that:

$$[a, [b, C]] = [[a, b], C] + (-1)^{p(a)p(b)} [b, [a, C]], \quad \text{for } a, b \in \mathfrak{g}_0, C \in \mathfrak{g}_1. \quad (5.5)$$

It follows from (5.5), (5.4) and (4.2), that

$$[[L_a, L_b], [L_c, P]] = (-1)^{p(c)p(b)} [L_{a \circ (c \circ b) - (a \circ c) \circ b}, P]. \quad (5.6)$$

Relations (5.3), (5.4) and (5.6) show that $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$. Finally, using identity (4.2), one verifies that \mathfrak{g}_0 is a subalgebra of $Lie(J)$. Therefore $Lie(J)$ is closed under the defined bracket. Besides, a direct computation, together with relations (5.1)–(5.4), shows that the Jacobi identity is satisfied. \square

Definition 5.2. A short grading of a Lie superalgebra \mathfrak{g} is a \mathbb{Z} -grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Definition 5.3. A short subalgebra of a Lie superalgebra \mathfrak{g} is an sl_2 subalgebra spanned by even elements e, h, f such that:

- (i) the eigenspace decomposition of $ad h$ defines a short grading of \mathfrak{g} ;
- (ii) $[h, e] = -e, [h, f] = f, [e, f] = h$.

Example 5.4. Let J be a Jordan superalgebra with unit element e and let $Lie(J)$ be the Lie superalgebra associated to J . Consider the elements $h_J := -L_e$ and $f_J = P$ in $Lie(J)$. Then the subalgebra \mathfrak{a}_J spanned by e , f_J and h_J is a short subalgebra of $Lie(J)$.

Lemma 5.5. *The adjoint representation of a short subalgebra \mathfrak{a} of a Lie superalgebra \mathfrak{g} exponentiates to the action of SL_2 by continuous automorphisms on \mathfrak{g} .*

Proof. By Definition 5.3, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. It follows that $(ade)^3 = 0$ and $(adf)^3 = 0$. Elements e and f thus exponentiate to continuous automorphisms of \mathfrak{g} since $\exp(ade) = 1 + ade + \frac{1}{2}(ade)^2$ and $\exp(adf) = 1 + adf + \frac{1}{2}(adf)^2$. Since \mathfrak{a} is generated by e and f , the thesis follows. \square

Definition 5.6. A \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (and the corresponding grading) is called minimal if any non-trivial ideal I of \mathfrak{g} intersects \mathfrak{g}_{-1} non-trivially, i.e., $I \cap \mathfrak{g}_{-1}$ is neither 0 nor \mathfrak{g}_{-1} .

If \mathfrak{g} is a Lie superalgebra and \mathfrak{a} is a short subalgebra of \mathfrak{g} such that the corresponding short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is minimal, then the pair $(\mathfrak{g}, \mathfrak{a})$ is called a minimal pair.

Lemma 5.7. *A \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is minimal if and only if the following three conditions hold:*

$$\text{if } [a, \mathfrak{g}_{-1}] = 0 \text{ for some } a \in \mathfrak{g}_0 \oplus \mathfrak{g}_1, \text{ then } a = 0 \quad (\text{transitivity}); \quad (5.7)$$

$$[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0; \quad (5.8)$$

$$[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1. \quad (5.9)$$

Proof. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a minimal grading and let $A = \{a \in \mathfrak{g}_0 \oplus \mathfrak{g}_1 \mid [a, \mathfrak{g}_{-1}] = 0\}$. If $A \neq 0$, then $I = \sum_{n \geq 0} (ad \mathfrak{g}_0)^n(A) + \sum_{n \geq 0} (ad \mathfrak{g}_0)^n([\mathfrak{g}_1, A]) \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a non-trivial ideal of \mathfrak{g} with $I \cap \mathfrak{g}_{-1} = 0$, hence contradicting the minimality of the grading. Transitivity thus follows. Likewise, if $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subsetneq \mathfrak{g}_0$, then $I = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ is a non-trivial ideal of \mathfrak{g} with $I \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$, thus contradicting the hypotheses. Finally, if $[\mathfrak{g}_0, \mathfrak{g}_1] \subsetneq \mathfrak{g}_1$, then $I = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_1]$ is a non-trivial ideal of \mathfrak{g} such that $I \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$, thus contradicting the minimality of the grading.

Conversely, assume that conditions (5.7)–(5.9) hold and let I be a non-trivial ideal of \mathfrak{g} . Then $I \cap \mathfrak{g}_{-1} \neq 0$ by condition (5.7), and $I \cap \mathfrak{g}_{-1} \neq \mathfrak{g}_{-1}$ by conditions (5.8) and (5.9). \square

Lemma 5.8. *Let J be a Jordan superalgebra and let $Lie(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding Lie superalgebra. Then condition (5.7) holds, provided that $L_a = 0$ only for $a = 0$, and condition (5.8) holds. If, in addition, J is unital, then also condition (5.9) holds, hence $(Lie(J), \mathfrak{a}_J)$ is a minimal pair.*

Proof. (5.7) follows immediately from the definition of the bracket in $Lie(J)$ and (5.8) follows from properties (5.1) and (5.2). Finally, if e is the unit element of J , then $[L_e, P] = -P$ which proves (5.9). Hence, by Lemma 5.7, if J is unital, then $(Lie(J), \mathfrak{a}_J)$ is a minimal pair. \square

Remark 5.9. If a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ comes from a short subalgebra, then it is minimal if and only if \mathfrak{g}_0 contains no non-zero ideals of \mathfrak{g} and $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

Proposition 5.10. *A unital Jordan superalgebra J is simple if and only if $\text{Lie}(J)$ is a simple Lie superalgebra.*

Proof. By Lemma 5.8, $\text{Lie}(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ satisfies (5.7)–(5.9). Suppose that J is simple. Then \mathfrak{g}_{-1} is an irreducible \mathfrak{g}_0 -module. Let $I \neq 0$ be an ideal of $\text{Lie}(J)$. Then, by (5.7), $I \cap \mathfrak{g}_{-1} \neq 0$, hence, by the irreducibility of \mathfrak{g}_{-1} , $I \supset \mathfrak{g}_{-1}$.

Then the simplicity of $\text{Lie}(J)$ follows from (5.8) and (5.9).

Conversely, if $\text{Lie}(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is simple, then \mathfrak{g}_{-1} is an irreducible \mathfrak{g}_0 -module. Indeed, if \mathfrak{h}_{-1} is a proper \mathfrak{g}_0 -submodule of \mathfrak{g}_{-1} , then $\mathfrak{h}_{-1} \oplus [\mathfrak{h}_{-1}, \mathfrak{g}_1] \oplus [[\mathfrak{h}_{-1}, \mathfrak{g}_1], \mathfrak{g}_1]$ is a proper ideal of $\text{Lie}(J)$. If $Y \neq 0$ is an ideal of $J = \mathfrak{g}_{-1}$, then Y is \mathfrak{g}_0 -stable, hence $Y = J$ by the irreducibility of \mathfrak{g}_{-1} . \square

If J is a non-unital Jordan superalgebra, we define $\tilde{J} = \mathbb{F}1 \oplus J$ and we extend the product of J to a product on \tilde{J} by setting $1 \circ x = x$. Then the following proposition holds.

Proposition 5.11. *If J is a simple non-unital Jordan superalgebra, then $\text{Lie}(\tilde{J}) \simeq S \rtimes \mathfrak{sl}_2$, where S is a simple Lie superalgebra and \mathfrak{sl}_2 acts on S by outer derivations. If J is a non-unital Jordan superalgebra such that $\text{Lie}(\tilde{J})$ is as above, then J is simple.*

Proof. By relations (5.1) and (5.3), elements 1 , $-L_1$ and P span a short subalgebra \mathfrak{a} of $\text{Lie}(\tilde{J})$. Let $\text{Lie}(\tilde{J}) = \text{Lie}(\tilde{J})_{-1} \oplus \text{Lie}(\tilde{J})_0 \oplus \text{Lie}(\tilde{J})_1$ be the corresponding short grading. By relations (5.2)–(5.4), $\text{Lie}(\tilde{J}) = S \rtimes \mathfrak{a}$, where $S = S_{-1} \oplus S_0 \oplus S_1$, $S_{-1} = J$, $S_1 = \langle [P, L_a] \mid a \in J \rangle$, and S_0 is the subalgebra of $\text{Lie}(\tilde{J})_0$ spanned by L_a and $[L_a, L_b]$ with a, b in J . Here \mathfrak{a} acts on S by outer derivations. The element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ exchanges $\text{Lie}(\tilde{J})_{-1}$ with $\text{Lie}(\tilde{J})_1$, hence it exchanges S_{-1} with S_1 , but the S_0 -module S_{-1} is irreducible since J is simple, hence $[S_0, S_{-1}] = S_{-1}$ (by definition, a simple algebra is an algebra with non-trivial product and no non-trivial ideals). It follows that $[S_0, S_1] = S_1$. Now the simplicity of S is proved as the simplicity of $\text{Lie}(J)$ in Proposition 5.10. The second part is proved as in Proposition 5.10 too. \square

Corollary 5.12. *Let J be a Jordan superalgebra such that $L_a = 0$ only if $a = 0$. Then $\text{Lie}(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a minimal \mathbb{Z} -graded superalgebra.*

Proof. If J is unital then the statement holds by Lemma 5.8. Now suppose that J is not unital. By Lemma 5.7, $\text{Lie}(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a minimal \mathbb{Z} -graded superalgebra if and only if it satisfies properties (5.7)–(5.9). Properties (5.7), (5.8) hold by Lemma 5.8, and property (5.9) holds by the proof of Proposition 5.11. \square

Let \mathcal{J} denote the category of unital Jordan superalgebras in which morphisms are Jordan superalgebra epimorphisms, and let \mathcal{L} denote the category of minimal pairs $(\mathfrak{g}, \mathfrak{a})$, where \mathfrak{g} is a Lie superalgebra, \mathfrak{a} is a short subalgebra of \mathfrak{g} , and a morphism φ from a pair $(\mathfrak{g}, \mathfrak{a})$ to a pair $(\mathfrak{g}', \mathfrak{b})$ is a Lie superalgebra epimorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $\varphi(\mathfrak{a}) = \mathfrak{b}$. According to Lemma 5.8, we construct a functor \mathcal{F} from \mathcal{J} to \mathcal{L} by associating to every unital Jordan superalgebra J the pair $(\text{Lie}(J), \mathfrak{a}_J)$, and to every epimorphism $\varphi: J \rightarrow J'$ of unital Jordan superalgebras the map $\varphi_{\mathcal{F}}: \text{Lie}(J) \rightarrow \text{Lie}(J')$ defined as follows:

$$x \mapsto \varphi(x), \quad L_x \mapsto L_{\varphi(x)} \quad \text{for every } x \in J; \quad P \mapsto P',$$

where P' is the bilinear map on J' defined by: $P'(a, b) = a \circ b$ for $a, b \in J'$.

One checks that $\varphi_{\mathcal{F}}$ is a Lie algebra epimorphism mapping \mathfrak{a}_J to $\mathfrak{a}_{J'}$. We will show that \mathcal{F} defines an equivalence of categories. In order to construct the inverse functor to \mathcal{F} we need the following result [14,23], [11, Lemma 4].

Lemma 5.13. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a \mathbb{Z} -graded Lie superalgebra and let $p \in (\mathfrak{g}_1)_{\bar{0}}$. Set*

$$x \circ y := [[p, x], y]. \quad (5.10)$$

Then the $\mathbb{Z}/2\mathbb{Z}$ -graded space \mathfrak{g}_{-1} with product (5.10) is a Jordan superalgebra.

Remark 5.14. Let \mathfrak{g} be a Lie superalgebra containing a short subalgebra \mathfrak{a} with standard generators $\{e, h, f\}$ as in Definition 5.3. Consider the corresponding short grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. Define on $J(\mathfrak{g}, \mathfrak{a}) := \mathfrak{g}_{-1}$ product (5.10), where $p = f$. Then, by Lemma 5.13, $J(\mathfrak{g}, \mathfrak{a})$ is a Jordan superalgebra and e is its unit element.

Theorem 5.15. *The functor \mathcal{F} is an equivalence of categories.*

Proof. In order to define the inverse functor to \mathcal{F} we associate to the minimal pair $(\mathfrak{g}, \mathfrak{a})$, where \mathfrak{g} is a Lie superalgebra and \mathfrak{a} is a short subalgebra of \mathfrak{g} , the Jordan superalgebra $J(\mathfrak{g}, \mathfrak{a})$ defined in Remark 5.14, and to every Lie superalgebra epimorphism $\psi : (\mathfrak{g}, \mathfrak{a}) \rightarrow (\mathfrak{g}', \mathfrak{b})$, the Jordan superalgebra epimorphism $\psi_{\Phi} : J(\mathfrak{g}, \mathfrak{a}) \rightarrow J(\mathfrak{g}', \mathfrak{b})$, $\psi_{\Phi} = \psi|_{\mathfrak{g}_{-1}}$. Notice that, since ψ is a Lie superalgebra epimorphism mapping \mathfrak{a} to \mathfrak{b} , due to Lemma 5.5, we may assume that the corresponding triples $\{e, h, f\}$ are mapped to each other, hence, by (5.10), ψ induces an epimorphism of Jordan superalgebras. Therefore we have defined a functor from \mathcal{L} to \mathcal{J} . Let us denote it by Φ .

Let $(\mathfrak{g}, \mathfrak{a})$ be a minimal pair, with $\mathfrak{a} = \langle e, f, h \rangle$ and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding minimal grading, and let $(Lie(J), \mathfrak{a}_J)$ be the pair associated to $J(\mathfrak{g}, \mathfrak{a})$ by \mathcal{F} . From the representation theory of $\mathfrak{a} \simeq sl_2$, applied to its adjoint action on \mathfrak{g} , it follows that: $[e, \mathfrak{g}_0] = \mathfrak{g}_{-1}$, $[f, \mathfrak{g}_0] = \mathfrak{g}_1$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, f] + \mathfrak{g}_0^f$, where \mathfrak{g}_0^f is the centralizer of f in \mathfrak{g}_0 . Since, by the minimality of $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$, it follows that $\mathfrak{g}_0 = [[\mathfrak{g}_0, f], \mathfrak{g}_{-1}] = [[[\mathfrak{g}_{-1}, f], f], \mathfrak{g}_{-1}] = [[\mathfrak{g}_{-1}, f], [\mathfrak{g}_{-1}, f]] + [\mathfrak{g}_{-1}, f]$, hence $(Lie(J), \mathfrak{a}_J) = (\mathfrak{g}, \mathfrak{a})$. Besides, if $\psi : (\mathfrak{g}, \mathfrak{a}) \rightarrow (\mathfrak{g}', \mathfrak{b})$ is a Lie superalgebra epimorphism such that $\psi(\mathfrak{a}) = \mathfrak{b}$, then $(\psi_{\Phi})_{\mathcal{F}} = \psi$.

Conversely, let J be a Jordan superalgebra with unit element e . Consider the pair $(Lie(J), \mathfrak{a}_J)$ associated to J by \mathcal{F} and the unital Jordan superalgebra $J(Lie(J), \mathfrak{a}_J)$ associated to $(Lie(J), \mathfrak{a}_J)$ by Φ . By construction, if a and b lie in $J(Lie(J), \mathfrak{a}_J)$, their product is $a \circ b = [[[P, L_e], a], b]$. By (5.2), this is the same as the product of a and b in J . Besides, if $\varphi : J \rightarrow J'$ is an epimorphism of unital Jordan superalgebras, and $\varphi_{\mathcal{F}} : (Lie(J), \mathfrak{a}_J) \rightarrow (Lie(J'), \mathfrak{a}_{J'})$ is the corresponding epimorphism of pairs in \mathcal{L} , with $Lie(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $\varphi_{\mathcal{F}|_{\mathfrak{g}_{-1}}} = \varphi$. We have thus shown that Φ is the inverse functor to \mathcal{F} . \square

Let \mathcal{J}_{lc} denote the subcategory of \mathcal{J} consisting of unital linearly compact Jordan superalgebras in which morphisms are continuous Jordan superalgebra epimorphisms, and let \mathcal{L}_{lc} denote the subcategory of \mathcal{L} consisting of minimal pairs $(\mathfrak{g}, \mathfrak{a})$, where \mathfrak{g} is a linearly compact Lie superalgebra, and morphisms from a pair $(\mathfrak{g}, \mathfrak{a})$ to a pair $(\mathfrak{g}', \mathfrak{b})$ are continuous Lie superalgebra epimorphisms $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $\varphi(\mathfrak{a}) = \mathfrak{b}$.

The TKK construction carries over to the case of a linearly compact J almost verbatim, except that in the construction of \mathfrak{g}_0 in $Lie(J)$ one takes the closure of the corresponding span.

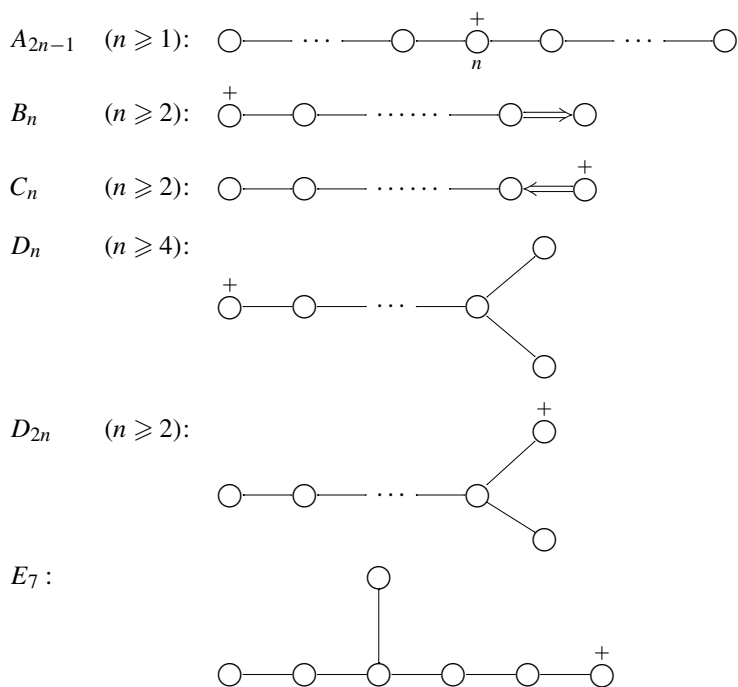
Proposition 5.16. *The functor \mathcal{F} is an equivalence of the categories \mathcal{J}_{lc} and \mathcal{L}_{lc} .*

Proof. \mathcal{F} is a functor from \mathcal{J}_{lc} to \mathcal{L}_{lc} due to Lemma 2.1. Besides, functor Φ maps \mathcal{L}_{lc} to \mathcal{J}_{lc} since if \mathfrak{g} is a linearly compact Lie superalgebra and \mathfrak{a} is a short subalgebra of \mathfrak{g} inducing on \mathfrak{g} the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then \mathfrak{g}_{-1} is a closed subspace of \mathfrak{g} , hence it is linearly compact. By construction, continuous epimorphisms are mapped to continuous epimorphisms. \square

6. Classification of short gradings of simple linearly compact Lie superalgebras

Let L be a simple finite-dimensional Lie algebra and choose a vertex i of its Dynkin diagram. Consider the \mathbb{Z} -grading of L defined by setting $\deg(e_i) = 1$, $\deg(f_i) = -1$ and the degrees of all other Chevalley generators of L equal to 0. We will denote such a grading by the Dynkin diagram of L where the i th vertex is labelled by $+$. The following well-known theorem describes all gradings defined by short subalgebras of all simple finite-dimensional Lie algebras.

Theorem 6.1. *A complete list, up to isomorphism, of short gradings defined by short subalgebras of all simple finite-dimensional Lie algebras is as follows:*



Proof. Any \mathbb{Z} -grading $L = \bigoplus_{j \in \mathbb{Z}} L_j$ of a simple finite-dimensional Lie algebra L is defined by letting $\deg(e_i) = -\deg(f_i) = k_i \in \mathbb{Z}_+$, where e_i, f_i are the Chevalley generators of L . Let $\sum_i a_i \alpha_i$ be the highest root of L . It is clear that a \mathbb{Z} -grading is short if and only if all k_i 's are 0, except for $k_s = 1$, such that $a_s = 1$. Next, if a short grading comes from a short subalgebra $\mathfrak{a} = \langle e, h, f \rangle$, then

$$(L_0^e, h) = 0, \quad (6.1)$$

where (\cdot, \cdot) is the Killing form of L and L_0^e denotes the centralizer of e in L_0 . Indeed: $(L_0^e, [e, f]) = ([L_0^e, e], f) = 0$. Conversely, if (6.1) holds, then $h \in [e, L_1]$, since $[e, L_1] \perp L_0^e$ and $L_0 = L_0^e + [e, L_1]$ (by the representation theory of sl_2). Thus (6.1) is a necessary and sufficient condition for a short grading to come from a short subalgebra, and this condition (along with $a_s = 1$) produces the above list. \square

Remark 6.2. We denote by $\mathfrak{a}(sl_{2n})$, $\mathfrak{a}'(so_{2n+1})$, $\mathfrak{a}(sp_{2n})$, $\mathfrak{a}'(so_{2n})$, $\mathfrak{a}(so_{4n})$, and $\mathfrak{a}(E_7)$, respectively, the short subalgebras of the simple Lie algebras corresponding to the diagrams listed in Theorem 6.1. If for the definition of the classical Lie algebras so_{2n+1} , sp_{2n} , and so_{2n} we choose the bilinear forms with matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

respectively, in some basis, then the elements h in the corresponding short subalgebras are the following:

$$\begin{aligned} \mathfrak{a}(sl_{2n}): \quad h &= \frac{1}{2} \text{diag}(I_n, -I_n); \\ \mathfrak{a}'(so_{2n+1}): \quad h &= \text{diag}(0, 1, 0, \dots, 0, -1, 0, \dots, 0); \\ \mathfrak{a}(sp_{2n}): \quad h &= \frac{1}{2} \text{diag}(I_n, -I_n); \\ \mathfrak{a}'(so_{2n}): \quad h &= \text{diag}(1, 0, \dots, 0, -1, 0, \dots, 0); \\ \mathfrak{a}(so_{4n}): \quad h &= \frac{1}{2} \text{diag}(I_{2n}, -I_{2n}). \end{aligned}$$

Theorem 6.3. A complete list, up to isomorphism, of pairs (S, \mathfrak{a}) where S is a simple linearly compact Lie superalgebra, and \mathfrak{a} is a short subalgebra of $\text{Der } S$, is as follows:

1. $S = sl(2m, 2n)/\delta_{m,n} \mathbb{F} I_{2m+2n}$, \mathfrak{a} is the diagonal subalgebra of $\mathfrak{a}(sl_{2m}) + \mathfrak{a}(sl_{2n})$;
2. $S = osp(m, 2r)$ with $m \geq 3$ and $(m, r) \neq (4, 0)$, $\mathfrak{a} = \mathfrak{a}'(so_m)$;
3. $S = osp(4l, 2r)$ with $(l, r) \neq (1, 0)$, \mathfrak{a} the diagonal subalgebra of $\mathfrak{a}(so_{4l}) + \mathfrak{a}(sp_{2r})$;
4. $S = p(2k)$ with $k > 1$, $\mathfrak{a} = \mathfrak{a}(sl_{2k})$;
5. $S = q(2k)$ with $k > 1$, $\mathfrak{a} = \mathfrak{a}(sl_{2k})$;
6. $S = E_7$, $\mathfrak{a} = \mathfrak{a}(E_7)$;
7. $S = F(4)$, \mathfrak{a} is the diagonal subalgebra of $\mathfrak{a}(sl_2) + \mathfrak{a}(so_7)$;
8. $S = D(2, 1; \alpha)$, \mathfrak{a} is the diagonal subalgebra of $\mathfrak{a}(sl_2) + \mathfrak{a}(sl_2)$;
9. $S = sl(2, 2)/\mathbb{F} I_4$, \mathfrak{a} is the subalgebra of outer derivations of $sl(2, 2)/\mathbb{F} I_4$ (see [10, Proposition 5.1.2(e)]);
10. $S = H(2k, n)$ with $n \geq 3$ and $(k, n) \neq (0, 3)$, $\mathfrak{a} = \langle \xi_n \xi_{n-1}, \xi_{n-2} \xi_n, \xi_{n-2} \xi_{n-1} \rangle$;
11. $S = K(2k+1, n)$ with $n \geq 3$, $\mathfrak{a} = \langle \xi_n \xi_{n-1}, \xi_{n-2} \xi_n, \xi_{n-2} \xi_{n-1} \rangle$;
12. $S = E(1, 6)$ and $\mathfrak{a} = \langle \xi_5 \xi_6, \xi_4 \xi_6, \xi_4 \xi_5 \rangle$;
13. $S = S(1, 2)$, \mathfrak{a} is the subalgebra of outer derivations of $S(1, 2)$ (see Example 6.7).

Proof. Let S be a simple linearly compact Lie superalgebra, let $\text{Der } S$ be the Lie superalgebra of its continuous derivations (recall that it is linearly compact artinian semisimple [3]), and let \mathfrak{a} be a short subalgebra with the basis e , h and f as in Definition 5.3, of the Lie superalgebra $\text{Der } S$. Then it is a short subalgebra of a maximal reductive subalgebra of $\text{Der } S$. Recall that

a finite-dimensional even subalgebra is called reductive if its adjoint representation on the whole Lie superalgebra decomposes into a product of finite-dimensional irreducible submodules, that all maximal reductive subalgebras of an artinian semisimple linearly compact Lie superalgebra ($Der S$ is the superalgebra in question) are conjugate to each other, and any reductive subalgebra is contained in one of them (see [3, Theorem 1.7]; this theorem is stated for tori, but its proof works for reductive subalgebras as well).

We will go over the list of all finite- and infinite-dimensional simple linearly compact Lie superalgebras S [10,12]. Let us first assume that $\mathfrak{a} \subset S$. If $S = sl(r, s)$ with $r \neq s$, then, since $S_0 = sl_r \oplus sl_s \oplus \mathbb{F}$, by Theorem 6.1, either $r = 2m$, $s = 2n$, and h is one of the following: $h_1 = 1/2 \text{diag}(I_m, -I_m, I_n, -I_n)$, $h_2 = 1/2 \text{diag}(I_m, -I_m, 0, 0)$, $h_3 = 1/2 \text{diag}(0, 0, I_n, -I_n)$, or $r = 2m$, $s = 1$, and $h = h_2$. Using the description of the odd roots of S given in [10, §2.5], one sees that adh_2 and adh_3 have non-integers eigenvalues on S_{-1} , hence $r = 2m$, $s = 2n$ and $h = h_1$. The same argument takes care of all other finite-dimensional S with S_0 reductive. For example, if $S = p(n)$ or $S = q(n)$, then $S_0 = sl_n$, hence S has, up to conjugation, at most one short subalgebra, and it exists if and only if n is even.

Now consider $S = W(m, n)$ or $S = S(m, n)$ with the \mathbb{Z} -grading defined by $\deg(x_i) = -\frac{\partial}{\partial x_i} = 1$; $\deg(\xi_j) = -\deg(\frac{\partial}{\partial \xi_j}) = 1$. With this grading $S = \prod_{j \geq -1} S_j$ where $S_0 \simeq gl(m, n)$ if $S = W(m, n)$ and $S_0 \simeq sl(m, n)$ if $S = S(m, n)$. Then, \mathfrak{a} is conjugate to a short subalgebra of $sl(m, n)$, hence $r = 2m$, $s = 2n$, and $h = h_1$. Since S_{-1} is isomorphic to the standard S_0 -module, it follows that h has non-integer eigenvalues on S_{-1} . Hence $W(m, n)$ and $S(m, n)$ have no short subalgebras. Similar arguments show that $\tilde{S}(0, n)$, $HO(n, n)$, $SHO(n, n)$, $KO(n, n+1)$, $SKO(n, n+1; \beta)$, $SHO^{\sim}(n, n)$ and $SKO^{\sim}(n, n+1)$ have no short subalgebras.

Let $S = H(2k, n)$ and let $\mathfrak{a} = \langle e, h, f \rangle$ be a short subalgebra of S . Consider S with the \mathbb{Z} -grading defined by $S_j = \{P(x, \xi) \mid \deg P = j + 2\}$. Then $S = \prod_{j \geq -1} S_j$ where $S_0 \simeq osp(n, 2k)$ and S_{-1} is isomorphic to the standard $osp(n, 2k)$ -module. Therefore \mathfrak{a} is conjugate to a short subalgebra of $osp(n, 2k)$ such that adh has integer eigenvalues on S_{-1} . It follows that $n \geq 3$ and \mathfrak{a} is conjugate to $\mathfrak{a}'(so_n)$, i.e., up to conjugation, $\mathfrak{a} = \langle \xi_n \xi_{n-1}, \xi_{n-2} \xi_n, \xi_{n-2} \xi_{n-1} \rangle$. Similar arguments show that $K(m, n)$ with $n \geq 3$, and $E(1, 6)$ have, up to conjugation, a unique short subalgebra, and that $K(m, n)$ has no short subalgebras if $n \leq 2$.

If $S = E(5, 10)$, then the maximal reductive subalgebra of S is isomorphic to sl_5 , hence, by Theorem 6.1, S has no short subalgebras.

We recall that a \mathbb{Z} -grading of a Lie superalgebra L is equivalent to the choice of an ad-diagonalizable element t with integer eigenvalues in $Der L$. If L is an artinian semisimple linearly compact Lie superalgebra, then all maximal tori are conjugate by an inner automorphism of L [3, Theorem 1.7]. It follows that, up to conjugation by an inner automorphism of L , the \mathbb{Z} -gradings of L are parameterized by the set $P^\vee = \{t \in T \mid ad(t) \text{ has only integer eigenvalues}\}$, where T is a fixed maximal torus of L .

Let $S = E(3, 6)$. It follows from the above discussion and [6] that all \mathbb{Z} -gradings of S are parameterized by quadruples $(a_1, a_2, a_3, \varepsilon)$ such that $\varepsilon + \frac{1}{2} \sum_{i=1}^3 a_i \in \mathbb{Z}$, where:

$$\deg(x_i) = -\deg\left(\frac{\partial}{\partial x_i}\right) = a_i \in \mathbb{Z}, \quad \deg(d) = -\frac{1}{2} \sum_{i=1}^3 a_i, \quad (6.2)$$

$$\deg(e_1) = -\deg(e_2) = \varepsilon \in \frac{1}{2}\mathbb{Z}, \quad (6.3)$$

$$\deg(E) = -\deg(F) = 2\varepsilon. \quad (6.4)$$

It is immediate to see that a necessary condition for such a grading to be short (in fact, to be finite) is that $a_i = 0$ for every $i = 1, 2, 3$. But this implies $\varepsilon \in \mathbb{Z}$, and (6.4) shows that the resulting grading cannot be short. It follows that $E(3, 6)$ has no short subalgebras. Similar arguments, using the explicit description of \mathbb{Z} -gradings [3], show that the Lie superalgebras $E(3, 8)$ and $E(4, 4)$ have no short gradings, hence no short subalgebras.

According to [10, Proposition 5.1.2(e)], [12, Proposition 6.1] and its corrected version [3, Proposition 1.8], if S is a simple linearly compact Lie superalgebra such that $\text{Der } S$ contains a copy of sl_2 of outer derivations of S , then S is isomorphic to $sl(2, 2)/\mathbb{F}I_4$, $S(1, 2)$, $SHO(3, 3)$, or $SKO(2, 3; 1)$. One can show directly that these sl_2 's in $S = sl(2, 2)/\mathbb{F}I_4$ and $S = S(1, 2)$ (see Example 6.7) are short subalgebras of $\text{Der } S$. On the contrary, if $S = SHO(3, 3)$ or $S = SKO(2, 3; 1)$, then $\text{Der } S$ has no short gradings, hence it has no short subalgebras. Indeed, every \mathbb{Z} -grading of $SHO(3, 3)$ is, up to conjugation, the \mathbb{Z} -grading defined by

$$\deg(x_i) = -\deg\left(\frac{\partial}{\partial x_i}\right) = a_i \in \mathbb{Z}; \quad \deg(\xi_j) = -\deg\left(\frac{\partial}{\partial \xi_j}\right) = b_j \in \mathbb{Z}$$

for some a_i 's, b_j 's such that $a_i + b_i = c \in \mathbb{F}$. A necessary condition for such a grading to be short is that $a_i = 0$ for every $i = 1, 2, 3$, hence $b_1 = b_2 = b_3 = 1$ or -1 . But the induced grading of $\text{Der } S$ is not short (for a description of S and $\text{Der } S$ see [12]). Similar arguments show that $\text{Der } SKO(2, 3; 1)$ has no short subalgebras. \square

Remark 6.4. Let (S, α) be one of the pairs listed in Theorem 6.3, with S a simple finite-dimensional Lie superalgebra different from $H(0, n)$ (which is treated in Example 6.5), and let $L = S$ or $L = \text{Der } S$. Then the corresponding Jordan superalgebras $J(L, \alpha)$ are as follows:

1. $L = sl(2m, 2n)/\delta_{m,n}\mathbb{F}I_{2m+2n}$, $J(L, \alpha) = gl(m, n)_+$;
2. $L = osp(m, 2r)$, with $m \geq 3$, $(m, r) \neq (4, 0)$, $J(L, \alpha) = (m - 3, 2r)_+$;
3. $L = osp(4l, 2r)$, with $(l, r) \neq (1, 0)$, $J(L, \alpha) = osp(r, 2l)_+$;
4. $L = p(2k)$ with $k > 1$, $J(L, \alpha) = p(k)_+$;
5. $L = q(2k)$ with $k > 1$, $J(L, \alpha) = q(k)_+$;
6. $L = E_7$, $J(L, \alpha) = E$;
7. $L = F(4)$, $J(L, \alpha) = F$;
8. $L = D(2, 1; \alpha)$, $J(L, \alpha) = D_t$;
9. $L = \text{Der}(sl(2, 2)/\mathbb{F}I_4)$, $J(L, \alpha) = K$.

Example 6.5. Consider the simple Lie superalgebra $H = H(2k, n)$, where we assume that $n \geq 3$ if $k > 0$ and $n \geq 4$ if $k = 0$ (recall that $H(0, n)$ is simple iff $n \geq 4$). Then the subalgebra α of H spanned by elements $h = \xi_n \xi_{n-1}$, $e = \xi_{n-2} \xi_n$ and $f = \xi_{n-2} \xi_{n-1}$ is a short subalgebra of H whose adjoint action induces on H the grading $H = H_{-1} \oplus H_0 \oplus H_1$ defined by

$$\begin{aligned} \deg(p_i) = \deg(q_i) = 0, \quad i = 1, \dots, k; \\ \deg(\xi_j) = 0, \quad j = 1, \dots, n-2, \quad \deg(\xi_{n-1}) = 1, \quad \deg(\xi_n) = -1. \end{aligned} \quad (6.5)$$

Then $J(H, \mathfrak{a}) \simeq \mathcal{O}(2k, n-2)$ with reversed parity, with the following product: for f, g in $J(H, \mathfrak{a})$,

$$f \circ g = (-1)^{p(f)} \frac{\partial f}{\partial \xi_{n-2}} g + f \frac{\partial g}{\partial \xi_{n-2}} + (-1)^{p(f)} \xi_{n-2} \{f, g\}, \quad (6.6)$$

where $\{\cdot, \cdot\}$ is the restriction of bracket (3.2) to $\mathcal{O}(2k, n-2)$. By Propositions 5.10 and 5.16, $J(H, \mathfrak{a})$ is a simple linearly compact Jordan superalgebra. One easily checks, using (6.6), that the map

$$\begin{aligned} J(H, \mathfrak{a}) &\longrightarrow JP(2k, n-3), \\ \xi_{n-2}f + g &\longmapsto f + \eta g, \quad \text{for } f, g \in \mathcal{O}(2k, n-3), \end{aligned}$$

is an isomorphism of linearly compact Jordan superalgebras (see Example 4.3).

Example 6.6. Consider the simple Lie superalgebra $L = K(2k+1, n)$, where we assume that $n \geq 3$. Then the subalgebra \mathfrak{a} spanned by elements $h = \xi_n \xi_{n-1}$, $e = \xi_{n-2} \xi_n$ and $f = \xi_{n-2} \xi_{n-1}$ is a short subalgebra of L whose adjoint action induces on L the grading $L = L_{-1} \oplus L_0 \oplus L_1$ defined by (6.5) and the condition $\deg(t) = 0$. Then $J(L, \mathfrak{a}) \simeq \mathcal{O}(2k+1, n-2)$ with reversed parity, with the following product ($f, g \in J(L, \mathfrak{a})$):

$$f \circ g = (-1)^{p(f)} \left(\frac{\partial f}{\partial \xi_{n-2}} g + \{\xi_{n-2}f, g\} + \xi_{n-2}(1-E)(f) \frac{\partial g}{\partial t} - \xi_{n-2} \frac{\partial f}{\partial t} (1-E)(g) \right), \quad (6.7)$$

where $\{\cdot, \cdot\}$ is the restriction of bracket (3.2) to $\mathcal{O}(2k+1, n-2)$. By Propositions 5.10 and 5.16, $J(L, \mathfrak{a})$ is a simple linearly compact Jordan superalgebra. One easily checks, using (6.7), that the map

$$\begin{aligned} J(L, \mathfrak{a}) &\longrightarrow JP(2k+1, n-3), \\ \xi_{n-2}P + Q &\longmapsto P + \eta Q, \quad \text{for } P, Q \in \mathcal{O}(2k+1, n-3), \end{aligned}$$

is an isomorphism of linearly compact Jordan superalgebras (see Example 4.3).

Example 6.7. Let

$$S'(1, 2) = \{X \in W(1, 2) \mid \operatorname{div}(X) = 0\},$$

where for $X = P(x, \xi) \frac{\partial}{\partial x} + \sum_{i=1}^2 Q_i(x, \xi) \frac{\partial}{\partial \xi_i}$, $\operatorname{div}(X) = \frac{\partial P}{\partial x} + \sum_{i=1}^2 (-1)^{p(Q_i)} \frac{\partial Q_i}{\partial \xi_i}$, and let $S = S(1, 2) = [S'(1, 2), S'(1, 2)]$. Then $S'(1, 2) = S(1, 2) + \mathbb{F} \xi_1 \xi_2 \frac{\partial}{\partial x}$ and $S(1, 2)$ is simple [10, Example 4.2]. Besides, by [12, Proposition 6.1], $\operatorname{Der} S = S \rtimes \mathfrak{a}$ where $\mathfrak{a} \simeq \mathfrak{sl}_2$ acts on S by outer derivations and is spanned by $f = \xi_1 \xi_2 \frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial \xi_2}$, $h = \xi_1 \frac{\partial}{\partial \xi_1}$ and $e = -1/2(E + \xi_2 \frac{\partial}{\partial \xi_1})$, with E defined as follows: for $R = R(x)$,

$$\begin{aligned} E \left(R \xi_2 \frac{\partial}{\partial x} - \frac{\partial R}{\partial x} \xi_1 \xi_2 \frac{\partial}{\partial \xi_1} \right) &= -R \frac{\partial}{\partial \xi_1}, & E \left(R \xi_1 \frac{\partial}{\partial x} + \frac{\partial R}{\partial x} \xi_1 \xi_2 \frac{\partial}{\partial \xi_2} \right) &= R \frac{\partial}{\partial \xi_2}, \\ E \left(R \frac{\partial}{\partial x} - \frac{1}{2} \frac{\partial R}{\partial x} \left(\xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} \right) \right) &= 0, & \text{and } E(R(x)S(0, 2)) &= 0. \end{aligned}$$

Notice that \mathfrak{a} is a short subalgebra of $\text{Der } S$ whose action induces on S the \mathbb{Z} -grading defined by

$$\deg x = -\deg \frac{\partial}{\partial x} = 0, \quad \deg \xi_1 = -\deg \frac{\partial}{\partial \xi_1} = 1, \quad \deg \xi_2 = -\deg \frac{\partial}{\partial \xi_2} = 0.$$

Indeed, with this grading, $S = S_{-1} \oplus S_0 \oplus S_1$, where

$$\begin{aligned} S_{-1} &= \left\langle \frac{\partial}{\partial \xi_1} \right\rangle \otimes \mathbb{F}[[x]] \otimes \Lambda(\xi_2); \\ S_0 &= \left(\left\langle \frac{\partial}{\partial x}, \xi_1 \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \right\rangle \otimes \mathbb{F}[[x]] \otimes \Lambda(\xi_2) \right) \cap S; \\ S_1 &= \left(\left\langle \xi_1 \frac{\partial}{\partial x}, \xi_1 \frac{\partial}{\partial \xi_2} \right\rangle \otimes \mathbb{F}[[x]] \otimes \Lambda(\xi_2) \right) \cap S, \end{aligned}$$

hence it is immediate to check that $S_i = \{X \in S(1, 2) \mid [h, X] = iX\}$.

By Propositions 5.10 and 5.16, $J(S, \mathfrak{a})$ is a simple linearly compact Jordan superalgebra. An easy computation shows that $J(S, \mathfrak{a}) \simeq JS$ (cf. Example 4.6).

Example 6.8. The Lie superalgebra $L = E(1, 6)$ is a simple subalgebra of $K(1, 6)$ [12]. The subalgebra $\mathfrak{a} = \langle \xi_4 \xi_6, \xi_6 \xi_5, \xi_4 \xi_5 \rangle$ is a short subalgebra of $K(1, 6)$ contained in L , hence it defines on L the \mathbb{Z} -grading induced by the grading of $K(1, 6)$ described in Example 6.6 (for $n = 6$). It follows from [13] that $J(L, \mathfrak{a}) \simeq JCK$ (cf. Example 4.5).

7. Proof of Theorem 4.7 and a corollary

Proof of Theorem 4.7. By Propositions 5.10 and 5.16, J is a linearly compact unital simple Jordan superalgebra if and only if $(\text{Lie}(J), \mathfrak{a}_J)$ is a minimal pair, with $\text{Lie}(J)$ a simple linearly compact Lie superalgebra. Besides, by Proposition 5.11, if J is a non-unital simple Jordan superalgebra, then $(\text{Lie}(\tilde{J}), \mathfrak{a}_{\tilde{J}})$ is a minimal pair with $\text{Lie}(\tilde{J}) = S \rtimes \mathfrak{a}_{\tilde{J}}$, where S is simple and $\mathfrak{a}_{\tilde{J}}$ acts on S by outer derivations. The result then follows from Theorem 6.3, Remark 6.4 and Examples 6.5–6.8. \square

Corollary 7.1.

- (a) Let $(A, \{\cdot, \cdot\})$ be a unital simple linearly compact generalized Poisson superalgebra. Then the Jordan superalgebra $J(A)$ is isomorphic to one of the superalgebras $JP(m, n)$.
- (b) Any unital simple linearly compact generalized Poisson superalgebra is gauge equivalent to one of the generalized Poisson superalgebras $P(m, n)$.
- (c) Any unital simple linearly compact Poisson superalgebra is isomorphic to a Poisson superalgebra $P(m, n)$ with m even.

Proof. By Remark 4.4, $(A, \{\cdot, \cdot\}_D)$ is simple, hence, by [16, Theorem 1], its KKM double $J(A)$ is simple. Furthermore, by construction, $J(A)$ is unital and admits an involution such that its eigenspace decomposition $J(A) = J(A)_1 \oplus J(A)_{-1}$ has the properties that $J(A)_1$ is a unital associative subalgebra and the right multiplication by its elements induces a $J(A)_1$ -module

structure on $J(A)_{-1}$, isomorphic to the right regular $J(A)_1$ -module with reversed parity. Now it is easy to deduce from Theorem 4.7 that $J(A)$ is isomorphic to one of the Jordan superalgebras $JP(m, n)$, proving (a).

Since, in the case when A is purely even, in the above discussion $J(A)_1$ (respectively $J(A)_{-1}$) is the even (respectively odd) part of $J(A)$, (b) in the non-super case follows immediately as well. In the super case we use the description of the involutions of $L = \text{Lie}(J(A)) \simeq H(m, n)$ or $K(m, n)$, in order to conclude that every involution of $J(A)$ is conjugate to one that changes signs of indeterminates (it follows from [4] that any involution of L is conjugate to one of the subgroup $\text{Autr } L \simeq \mathbb{F}^\times (Sp_{2[m/2]} \times O_n)$ of $\text{Aut } L$).

Due to the associativity of $J(A)_1$, if the elements ηa_i lie in $J(A)_1$, $i = 1, 2, 3$, then

$$\{a_1, a_2\}a_3 = a_1\{a_2, a_3\}. \quad (7.1)$$

We check, using condition (7.1), that $J(A)_1 = A$ and $J(A)_{-1} = \eta A$ for all $A = P(m, n)$, except for $A = P(0, 1)$, in which case there is one other involution of $J(A)$ with required properties, but it is conjugate to the standard one by the permutation of ξ_1 and η . This proves (b). Claim (c) follows easily from (b). \square

A classification of local transitive Lie algebras on a finite-dimensional manifold in terms, analogous to those of Corollary 7.1(b), was obtained in [17]. The finite-dimensional case of Corollary 7.1(c) was obtained in [15] (the argument there is not quite correct).

Remark 7.2. Gauge equivalence classes of generalized Poisson brackets on a unital associative commutative algebra A are in a canonical bijective correspondence with Jordan superalgebras J , such that $J_0 \simeq A$, and the right multiplication by elements of J_0 defines an A -module structure on J_1 , isomorphic to the right regular A -module.

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Appendix A. All known infinite-dimensional simple Jordan superalgebras over \mathbb{F}

Examples A.1. All known examples of infinite-dimensional simple Jordan superalgebras over an algebraically closed field \mathbb{F} of characteristic zero, are of the following type:

1. A^+ , where A is a simple associative superalgebra, with Jordan product

$$a \circ b = \frac{1}{2}(ab + (-1)^{p(a)p(b)}ba).$$

2. $BCP(A, \omega)$: the fixed point subalgebra of A^+ under a super anti-involution ω .
3. $D(V, (\cdot, \cdot))$: the Jordan superalgebra of a non-degenerate super-symmetric bilinear form.
4. $Q(A) = A \oplus \bar{A}$, where A is a simple associative algebra, \bar{A} a copy of A with odd parity, with Jordan product

$$(a \oplus b) \circ (a_1 \oplus b_1) = (aa_1 + bb_1, ab_1 + ba_1).$$

5. $J(P)$: the KKM double of a simple generalized Poisson superalgebra P .
6. $JS(A, D)$, where A is a commutative associative superalgebra with an odd derivation D , such that A contains no non-trivial D -invariant ideals, with reversed parity and Jordan product

$$a \circ b = aD(b) + (-1)^{p(a)}D(a)b.$$

(The example $JS(\Lambda(1), d/d\xi)$ shows that these Jordan superalgebras are not always simple.)

7. $JCK(A, D)$, where A is a commutative associative algebra with an even derivation D , such that A contains no non-trivial D -invariant ideals (see Example 4.5).

Question A.2. Are there any other examples?

Remark A.3. It is shown in [20,22] that if $\text{char } \mathbb{F} > 3$, all unital finite-dimensional simple Jordan superalgebras over \mathbb{F} are either of types 1–5, 7, or they are isomorphic to D_t , E , F . It is proved in [25] that all the non-unital ones are either $J(A, d)$ (hence of type 6) or isomorphic to K .

The following remarks explain why there are no infinite-dimensional linearly compact simple Jordan superalgebras of types 1–4.

Remarks A.4. (a) Any linearly compact associative superalgebra contains an ideal of finite codimension. Hence the simple ones are finite-dimensional.

(b) A non-degenerate bilinear form exists on a linearly compact space V (respectively $\text{Hom}(V, V)$ is linearly compact) if and only if $\dim V < \infty$.

Appendix B. The finite-dimensional Jordan superalgebras D_t , K and F

In this section we recall the construction of the exceptional finite-dimensional Jordan superalgebras D_t , K and F .

The superalgebra D_t ($t \in \mathbb{F}, t \neq 0$). This is a parametric series of unital four-dimensional Jordan superalgebras defined as follows: $(D_t)_0 = \mathbb{F}e_1 + \mathbb{F}e_2$, $(D_t)_1 = \mathbb{F}\xi + \mathbb{F}\eta$, with multiplication table:

$$e_i^2 = e_i, \quad e_1 \circ e_2 = 0, \quad e_i \circ \xi = \frac{1}{2}\xi, \quad e_i \circ \eta = \frac{1}{2}\eta, \quad \xi \circ \eta = e_1 + te_2.$$

D_t is simple if and only if $t \neq 0$.

The superalgebra K . This is the three-dimensional simple non-unital Jordan superalgebra $K = \langle a \rangle \oplus \langle \xi_1, \xi_2 \rangle$, with multiplication table:

$$a^2 = a, \quad a \circ \xi_i = \frac{1}{2}\xi_i, \quad \xi_1 \circ \xi_2 = a.$$

The superalgebra F . In [8,11] it is defined by a multiplication table. More recently, I. Sheshtakov, and G. Benkart and A. Elduque [2] proposed the following beautiful construction. Consider on K the supersymmetric bilinear form (\cdot, \cdot) defined by

$$(a, a) = \frac{1}{2}, \quad (\xi_1, \xi_2) = 1, \quad (a, \xi_i) = 0.$$

Then $F = \mathbb{F}1 + (K \otimes K)$, with the unit element 1 and the product

$$(a \otimes b) \circ (c \otimes d) = (-1)^{p(b)p(c)} \left(ac \otimes bd - \frac{3}{4}(a, c)(b, d)1 \right),$$

for $a, b, c, d \in K$.

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